Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 2, pp. 175–189, 2002

COMPLICATED STRUCTURES OF GALILEAN-INVARIANT CONSERVATION LAWS

S. K. Godunov and V. M. Gordienko

UDC 517.9:539.3

This paper continues previous investigation of Galilean-invariant equations of mathematical physics, which was begun with the use of the Clebsch–Gordan coefficients from the theory of products of the represented group $SO(3)$. Complex systems of conservation laws and thermodynamic identities are constructed. Concrete examples are given.

Introduction. The present paper is a continuation of [1], in which we are studying the thermodynamic structures of the Galilean-invariant equations of mathematical physics which contain the laws of conservation of mass, momentum, and energy and compensating entropy equations. In order that such equations reflect the contents of the classical thermodynamic relations, a generating function L which depends on the required functions, vector-functions, and temperature should be used in their formulation. This generating function plays the role of the thermodynamic potential of the medium described by the equations. The conservation laws generated by such thermodynamic potentials have been discussed in the literature since the 1960s (see [1] and references therein). Unfortunately, the simplest structure of equations described in [1] does not cover all well-known examples from classical mathematical physics, which have been studied extensively. Such examples are described in [2] and in the last chapter of the monograph [3]. In particular, the equations of nonlinear elasticity and magnetic hydrodynamics do not fit into the simplest structure. The more complicated constructions of conservation laws proposed in the present paper already include the above examples. The dissipative processes of diffusion, heat conduction, and viscous friction are not considered here, except for one example.

The notation used herein is the same as in [1]. As in [1], we use the theory of orthogonal representations of the rotation group $SO(3)$, confining ourselves to odd-dimensional one-valued representations of integer weights N.

Complicated systems of equations are constructed in Sec. 2 from the initial simplest systems described in [1] by adding special additional terms into the equations. These terms are chosen so that they do not violate the conservation laws. The choice of possible terms is based on the collection of identities of generalized vector calculus given in Sec. 1.

In Sec. 3, of all the complicated systems described, we consider only those systems for which we proved that they can be written in the form of symmetric hyperbolic equations. Examples are given in which the equations studied are replaced by overdetermined compatible systems of conservation laws. Here we use and develop the scheme that was partly described in [3, 4]. Sec. 4 contains concrete examples of equations that enter into the described class of complicated thermodynamically consistent (compatible) structures.

1. Generalized Vector Calculus. Just as in [1], we assume that rotations of the coordinate system transform each of the unknown vector-functions $\boldsymbol{q}^{(A)}$ by an irreducible representation of weight A of the group $SO(3)$. We specify a coordinate vector $\boldsymbol{x}$ of three-dimensional space by its Cartesian components x_{-1}, x_0, and x_1. A vector-function $\boldsymbol{q}^{(N)}$ has $2N+1$ real components $q_n^{(N)}$ ($n = -N, -N+1, \ldots, -1, 0, 1, \ldots, N-1$, and N). Rotations of the coordinate system are specified by an orthogonal matrix $\mathcal{P}$ with positive determinant ($\mathcal{P}^{\rm t}\mathcal{P} = I_3$ and $\det \mathcal{P} = +1$); in this case, $\boldsymbol{x}$ is replaced by $\hat{\boldsymbol{x}} = \mathcal{P}\boldsymbol{x}$. In such a rotation, the vector-function $\boldsymbol{q}^{(N)}$ is transformed into $\hat{\boldsymbol{q}}^{(N)} = \Omega^{(N)}(\mathcal{P})\boldsymbol{q}^{(N)}$ by means of a standard $(2N+1) \times (2N+1)$ matrix $\Omega^{(N)}(\mathcal{P})$ which implements the representation. By the definition of representations, $\Omega^{(N)}(I_3) = I_{2N+1}$ and $\Omega^{(N)}(\mathcal{P}_1 \cdot \mathcal{P}_2) = \Omega^{(N)}(\mathcal{P}_1) \cdot \Omega^{(N)}(\mathcal{P}_2)$.

Sobolev Institute of Mathematics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 2, pp. 3–21, March–April, 2002. Original article submitted June 28, 2001.

0021-8944/02/4302-0175 $27.00

A rotation $\mathcal{P}$ is usually specified by Euler's angles φ, θ, and ψ:

$$\mathcal{P} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Explicit formulas for the entries of the matrix $\Omega^{(N)}(\mathcal{P})$, which express these entries in terms of Euler's angles φ, θ, and ψ, are given at the end of our previous paper [1] and are justified in [5]. It was noted there that $\Omega^{(1)}(\mathcal{P}) \equiv \mathcal{P}$.

We formulate the basic statements of the theory of Kronecker products of irreducible representations of the rotation group. Let $\boldsymbol{p}^{(L)}$ and $\boldsymbol{r}^{(M)}$ be vectors of dimensions $2L+1$ and $2M+1$ which, in rotations, are transformed by irreducible representations of weights L and M. The Kronecker product of these vectors is a rectangular $(2L+1)\times(2M+1)$ matrix Π with the entries $\Pi_{lm} = p_l^{(L)} \cdot r_m^{(M)}$. This matrix can be treated as a vector of dimension $(2L+1)\times(2M+1)$.

In rotations $\hat{\boldsymbol{x}} = \mathcal{P}\boldsymbol{x}$, the vector Π transforms into $\hat{\Pi}$ by transformations that correspond to a certain representation of the rotation group. Such a representation is called the *Kronecker product of representations* of weights L and M. If we consider that the scalar product in the space of $(2L+1)\times(2M+1)$ matrices is given by the formula $(\Phi, \Pi) = \sum\limits_{l,m} \Phi_{lm}\Pi_{lm} = \operatorname{tr}(\Phi\Pi^{\mathrm{t}})$, then the Kronecker product of orthogonal representations is also orthogonal, but if neither of the weights L and M is equal to zero, then it turns out to be reducible. It can be decomposed into a direct sum of irreducible orthogonal representations of weights $|L-M|$, $|L-M|+1$, $|L-M|+2$, $\ldots$, $L+M-1$, and $L+M$. The decomposition is implemented with the use of the so-called Clebsch–Gordan coefficients. It is reasonable to arrange these coefficients as matrix entries of special matrices. We shall refer to these matrices as Clebsch–Gordan matrices. Each such $(2L+1)\times(2M+1)$ matrix $G^k_{K[L,M]}$ is composed of entries $G^{k[l,m]}_{K[L,M]}$, where the superscripts l and m ($-L \leqslant l \leqslant L$ and $-M \leqslant m \leqslant M$) specify the numbers of the row and column on whose intersection this entry stands. The indices K and k number the matrices. The subscript K ($|L-M| \leqslant K \leqslant L+M$) is the weight of an irreducible representation and the superscript k is the number of a matrix which is a canonical basis element in the $(2K+1)$-dimensional subspace of matrices that transform according to a representation of weight K ($-K \leqslant k \leqslant K$).

There is some arbitrariness in the choice of canonical bases. The basis used here ensures the equality

$$G^k_{K[L,M]} = (-1)^{K+L+M}\{G^k_{K[M,L]}\}^{\mathrm{t}}, \tag{1.1}$$

i.e., permutation of the subscripts in the square brackets leads to transposition of the Clebsch–Gordan matrix and in the case where the sum $K+L+M$ is odd, to inversion of the signs of all entries. It is worth noting that the conditions of orthonormalization

$$\operatorname{tr}\{G^k_{K[L,M]} \cdot [G^n_{N[L,M]}]^{\mathrm{t}}\} = \delta_{KN} \cdot \delta_{kn} \tag{1.2}$$

can also be written as

$$\sum_{l=-L,\, m=-M}^{l=L,\, m=M} G^{k[l,m]}_{K[L,M]} \cdot G^{n[l,m]}_{N[L,M]} = \delta_{KN}\delta_{kn}.$$

We give several useful equalities:

$$\begin{gathered} G^{0[\lambda,l]}_{0[L,L]} = \delta_{\lambda l}/\sqrt{2L+1}, \\ G^{k[l,m]}_{K[L,M]} = \sqrt{(2K+1)/(2M+1)}\, G^{m[k,l]}_{M[K,L]} = (-1)^{K+L+M}\sqrt{(2K+1)/(2L+1)}\, G^{l[k,m]}_{L[K,M]}. \end{gathered} \tag{1.3}$$

Writing the Kronecker product of the vectors $\boldsymbol{p}^{(L)}$ and $\boldsymbol{r}^{(M)}$ in the form of a linear combination of the basis Clebsch–Gordan matrices

$$[\boldsymbol{p}^{(L)} \times \boldsymbol{r}^{(M)}] = \sum_{K=|L-M|}^{K=L+M} \left(\sum_{k=-K}^{k=K} w_k^{(K)} G^k_{K[L,M]} \right),$$

it is reasonable to group the coefficients $w_k^{(K)}$ of this linear combination into vectors $\boldsymbol{w}^{(K)}$ of dimensions $2K+1$. By virtue of (1.2), the components of these vectors are calculated by the rule

$$w_k^{(K)} = \operatorname{tr}\{[\boldsymbol{p}^{(L)} \times \boldsymbol{r}^{(M)}] \cdot [G^k_{K[L,M]}]^{\mathrm{t}}\}. \tag{1.4}$$

A rotation $\hat{\boldsymbol{x}} = \mathcal{P}\boldsymbol{x}$ of the coordinate system transforms the vectors $\boldsymbol{p}^{(L)}$ and $\boldsymbol{r}^{(M)}$ by the transformations $\hat{\boldsymbol{p}}^{(L)} = \Omega^{(L)}(\mathcal{P})\boldsymbol{p}^{(L)}$ and $\hat{\boldsymbol{r}}^{(M)} = \Omega^{(M)}(\mathcal{P})\boldsymbol{r}^{(M)}$, and this induces transformations of the vectors $\hat{\boldsymbol{w}}^{(K)} = \Omega^{(K)}(\mathcal{P})\boldsymbol{w}^{(K)}$. These transformations correspond to irreducible representations of the corresponding weights. It is reasonable to introduce the notation

$$\boldsymbol{w}^{(K)} = [\boldsymbol{p}^{(L)} \times \boldsymbol{r}^{(M)}]^{(K)} \tag{1.5}$$

and call $[\boldsymbol{p}^{(L)} \times \boldsymbol{r}^{(M)}]^{(K)}$ the *vector product of weight* K of the vectors $\boldsymbol{p}^{(L)}$ and $\boldsymbol{r}^{(M)}$. We recall that $|L - M| \leqslant K \leqslant L + M$ or, which is the same, $K \leqslant L + M$, $L \leqslant M + K$, and $M \leqslant K + L$ (the triangle inequalities). The vector product of weight 0 differs only by a factor from the scalar product of vector multipliers, which should have the same dimension (same weight):

$$[\boldsymbol{p}^{(L)} \times \boldsymbol{r}^{(L)}]^{(0)} = \frac{1}{\sqrt{2L+1}}\,(\boldsymbol{p}^{(L)}, \boldsymbol{r}^{(L)}) = \frac{1}{\sqrt{2L+1}} \sum_{l=-L}^{l=L} p_l^{(L)} r_l^{(L)}. \tag{1.6}$$

The following rule of permutation of multipliers follows from (1.1):

$$[\boldsymbol{p}^{(L)} \times \boldsymbol{r}^{(M)}]^{(K)} = (-1)^{K+L+M} [\boldsymbol{r}^{(M)} \times \boldsymbol{p}^{(L)}]^{(K)}.$$

In a natural way we introduce a generalized mixed product of three vectors of dimensions $2K+1$, $2L+1$, and $2M+1$ that is invariant under rotations:

$$(\boldsymbol{u}^{(K)}, \boldsymbol{v}^{(L)}, \boldsymbol{w}^{(M)}) = [\boldsymbol{u}^{(K)} \times [\boldsymbol{v}^{(L)} \times \boldsymbol{w}^{(M)}]^{(K)}]^{(0)}. \tag{1.7}$$

This product is meaningful if the weights K, L, and M satisfy the triangle inequalities. The generalized mixed product of the same multipliers is defined for every their order. When two vectors are interchanged, this product is multiplied by $(-1)^{K+L+M}$:

$$(\boldsymbol{v}^{(L)}, \boldsymbol{u}^{(K)}, \boldsymbol{w}^{(M)}) = (\boldsymbol{u}^{(K)}, \boldsymbol{w}^{(M)}, \boldsymbol{v}^{(L)}) = (-1)^{K+L+M} (\boldsymbol{u}^{(K)}, \boldsymbol{v}^{(L)}, \boldsymbol{w}^{(M)}) = (\boldsymbol{w}^{(M)}, \boldsymbol{v}^{(L)}, \boldsymbol{u}^{(K)})$$

or, which is the same,

$$\begin{aligned} &[\boldsymbol{v}^{(L)} \times [\boldsymbol{u}^{(K)} \times \boldsymbol{w}^{(M)}]^{(L)}]^{(0)} = [\boldsymbol{u}^{(K)} \times [\boldsymbol{w}^{(M)} \times \boldsymbol{v}^{(L)}]^{(K)}]^{(0)} \\ &= (-1)^{K+L+M} [\boldsymbol{u}^{(K)} \times [\boldsymbol{v}^{(L)} \times \boldsymbol{w}^{(M)}]^{(K)}]^{(0)} = [\boldsymbol{w}^{(M)} \times [\boldsymbol{v}^{(L)} \times \boldsymbol{u}^{(K)}]^{(M)}]^{(0)}. \end{aligned} \tag{1.8}$$

The validity of the above equalities follows from (1.1) and (1.3). The properties of the Clebsch–Gordan coefficients used above follow from analysis of their generating function.

Unfortunately, in the literature on the theory of representations of the rotation group, we were able to find the theory of the Clebsch–Gordan coefficients only for unitary representations. Thus, we had to study the case of orthogonal representations by introducing the corresponding changes into the unitary theory. As a result, we have obtained the generating function for the coefficients used in this study and have justified all their properties necessary for us. We are planning to publish this study in a separate paper.

The generalized formulas of vector calculus necessary for our purposes include matrices $G^k_{K[L,M]}$ in which K, L, and M are one or another permutation of a triple $1, N, N-1$ or a triple $1, N, N$. We give all nonzero entries of the matrices $G^i_{1[N,N-1]}$ and the formulas relating these matrices to the matrices $G^j_{1[N,N]}$.

In permutation of the integer parameters K, L, and M, the matrices transform by the rules (1.1) and (1.3).

Below we give the nonzero entries $G^{i[n,m]}_{1[N,N-1]}$:

$$G^{1[\pm(k-1),\pm k]}_{1[N,N-1]} = G^{-1[\pm k,\mp(k-1)]}_{1[N,N-1]} = \frac{1}{2}\sqrt{\frac{3(N-1)(N-k+1)}{N(2N-1)(2N+1)}} \qquad (2 \leqslant k \leqslant N-1),$$

$$\pm G^{1[\pm k,\mp(k-1)]}_{1[N,N-1]} = G^{-1[\pm(k-1),\mp k]}_{1[N,N-1]} = \pm\sqrt{\frac{3(N+k)(N+k-1)}{N(2N-1)(2N+1)}} \qquad (2 \leqslant k \leqslant N-1),$$

$$G^{1[\pm N,\pm(N-1)]}_{1[N,N-1]} = \sqrt{\frac{3}{2(2N+1)}}, \qquad G^{-1[-1,0]}_{1[N,N-1]} = G^{1[1,0]}_{1[N,N-1]} = \sqrt{\frac{3(N+1)}{2(2N-1)(2N+1)}},$$

$$G^{-1[0,-1]}_{1[N,N-1]} = G^{1[0,1]}_{1[N,N-1]} = -\sqrt{\frac{3(N-1)}{2(2N-1)(2N+1)}}.$$

The matrices $G^i_{1[N,N]}$ are calculated by the formula

$$G^i_{1[N,N]} = \sqrt{N(2N+1)/(3(N+1))}\,[G^k_{1[N,N-1]}G^j_{1[N-1,N]} - G^j_{1[N,N-1]}G^k_{1[N-1,N]}],$$

where the superscripts i, j, and k run over one of the following triples: $(-1,0,1)$, $(0,1,-1)$, or $(1,-1,0)$.

In what follows, we shall assume that the vectors considered depend on the spatial coordinates x_{-1}, x_0, and x_1 of some parametric 3D space, i.e., we consider vector fields $\boldsymbol{f}^{(N)}(x_{-1},x_0,x_1)$. We introduce a vector operator — the gradient operator

$$\nabla \equiv \nabla^{(1)} = \Big(\frac{\partial}{\partial x_{-1}}, \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}\Big)^{\mathrm{t}}$$

and decompose the field of derivatives of the vector $\boldsymbol{f}^{(N)}$ into three fields which, under rotations of the coordinate space, transform according to irreducible representations of weights $N-1$, N, and $N+1$. Such a decomposition is implemented using the operator equalities

$$\boldsymbol{p}^{(N-1)} = [\nabla\times\boldsymbol{f}^{(N)}]^{(N-1)}, \qquad \boldsymbol{q}^{(N)} = [\nabla\times\boldsymbol{f}^{(N)}]^{(N)}, \qquad \boldsymbol{r}^{(N+1)} = [\nabla\times\boldsymbol{f}^{(N)}]^{(N+1)},$$

which generalize the well-known rules of the classical vector calculus concerning the action of divergence, curl, and gradient operators.

The operator equalities are written in coordinate form using the Clebsch–Gordan coefficients. This form formally extends definitions (1.4) to the operator case.

Using the vector gradient operator ∇ and formulas (1.8) for $M = 1$, we obtain the following formulas of invariant differentiation:

$$[\nabla\times[\boldsymbol{u}^{(K)}\times\boldsymbol{v}^{(L)}]^{(1)}]^{(0)} = [\boldsymbol{v}^{(L)}\times[\nabla\times\boldsymbol{u}^{(K)}]^{(L)}]^{(0)} + (-1)^{K+L+1}[\boldsymbol{u}^{(K)}\times[\nabla\times\boldsymbol{v}^{(L)}]^{(K)}]^{(0)}. \tag{1.9}$$

The weights K and L should satisfy the inequalities $|K-L|\leqslant 1$ and $K+L\geqslant 1$. In fact, formula (1.9) contains formulas of three types for $L=K-1$, $L=K$, and $L=K+1$.

We give one more formula of differentiation of the double product. In formula (1.9), replacing $\boldsymbol{u}^{(K)}$ by $[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}$ and $\boldsymbol{v}^{(L)}$ by $\boldsymbol{v}^{(1)}$, we obtain

$$[\nabla\times[[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}\times\boldsymbol{v}^{(1)}]^{(1)}]^{(0)} = [\boldsymbol{v}^{(1)}\times[\nabla\times[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}]^{(1)}]^{(0)}$$
$$+ (-1)^K[[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}\times[\nabla\times\boldsymbol{v}^{(1)}]^{(K)}]^{(0)}.$$

Using property (1.8), we transform the second term on the right side of the above equality:

$$[[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}\times[\nabla\times\boldsymbol{v}^{(1)}]^{(K)}]^{(0)} = [\boldsymbol{p}^{(L)}\times[\boldsymbol{q}^{(M)}\times[\nabla\times\boldsymbol{v}^{(1)}]^{(K)}]^{(L)}]^{(0)}.$$

We finally obtain

$$[\nabla\times[[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}\times\boldsymbol{v}^{(1)}]^{(1)}]^{(0)} = [\boldsymbol{v}^{(1)}\times[\nabla\times[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}]^{(1)}]^{(0)}$$
$$+ (-1)^K[\boldsymbol{p}^{(L)}\times[\boldsymbol{q}^{(M)}\times[\nabla\times\boldsymbol{v}^{(1)}]^{(K)}]^{(L)}]^{(0)}. \tag{1.10}$$

In (1.10), the weight K can take values $K = 0$, 1, and 2 and, moreover, the triple K, L, M should satisfy the triangle inequalities.

For the applications considered in Sec. 2, based on (1.6)–(1.8), it is convenient to write identities (1.9) and (1.10) with the use of the symbol of scalar product:

$$\frac{\partial}{\partial x_j}[\boldsymbol{u}^{(K)}\times\boldsymbol{v}^{(L)}]^{(1)}_j \equiv \sqrt{3}[\nabla\times[\boldsymbol{u}^{(K)}\times\boldsymbol{v}^{(L)}]^{(1)}]^{(0)}$$
$$= \sqrt{3/(2L+1)}\,(\boldsymbol{v}^{(L)},[\nabla\times\boldsymbol{u}^{(K)}]^{(L)}) + (-1)^{K+L+1}\sqrt{3/(2K+1)}\,(\boldsymbol{u}^{(K)},[\nabla\times\boldsymbol{v}^{(L)}]^{(K)}), \tag{1.11}$$

$$\frac{\partial}{\partial x_j}\,[[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}\times\boldsymbol{v}^{(1)}]^{(1)}_j = (\boldsymbol{v}^{(1)},[\nabla\times[\boldsymbol{p}^{(L)}\times\boldsymbol{q}^{(M)}]^{(K)}]^{(1)})$$
$$+ (-1)^K\sqrt{3/(2L+1)}\,(\boldsymbol{p}^{(L)},[\boldsymbol{q}^{(M)}\times[\nabla\times\boldsymbol{v}^{(1)}]^{(K)}]^{(L)}). \tag{1.12}$$

Along with the vector-functions $\boldsymbol{q}^{(N)}(x_{-1},x_0,x_1)$ transformed by irreducible representations, it is sometimes useful to consider tensor-functions of the second rank $q^{(1,N)}$ with the components $q^{(1,N)}_{in}(x_{-1},x_0,x_1)$ ($-1\leqslant i\leqslant 1$ and

$-N \leqslant n \leqslant N$). In this case, unlike in the adopted practice, the first and second subscripts can run over different sets of admissible values. A rotation $\hat{\boldsymbol{x}} = \mathcal{P}\boldsymbol{x}$ is associated with the transformation $\hat{q}_{in}^{(1,N)} = \sum\limits_{k,m} \Omega_{ik}^{(1)}(\mathcal{P})\Omega_{nm}^{(N)}(\mathcal{P})q_{km}^{(1,N)}$, i.e., the tensor-functions transform according to the product of representations of weights 1 and N. This representation is reducible and can be decomposed into a direct sum of irreducible representations of weights $N-1$, N, and $N+1$, but we shall not make this decomposition.

Proceeding to derivation of the necessary identities, we make use of three elementary equalities, which are checked directly:

$$\frac{\partial}{\partial x_k}(u_i q_i p_k) = q_i p_k \frac{\partial u_i}{\partial x_k} + u_i \frac{\partial (q_i p_k)}{\partial x_k}, \tag{1.13a}$$

$$\frac{\partial}{\partial x_k}(u_k q_i p_i) = u_k \frac{\partial}{\partial x_k}(q_i p_i) + q_i p_i \frac{\partial u_k}{\partial x_k}, \tag{1.13b}$$

$$\frac{\partial}{\partial x_k}(u_i q_k p_i) = q_k p_i \frac{\partial u_i}{\partial x_k} + u_i \frac{\partial (q_k p_i)}{\partial x_k}. \tag{1.13c}$$

Assuming that summation is made over pairs of the same indices $i, k = -1, 0, 1$, we leave equality (1.13a) unchanged and replace some umbral subscripts in (1.13b) and (1.13c):

$$\frac{\partial}{\partial x_k}(u_k p_i q_i) = u_i \frac{\partial}{\partial x_k}(\delta_{ik} q_j p_j) + q_i\Big(p_i \frac{\partial u_k}{\partial x_k}\Big), \tag{1.13b$'$}$$

$$\frac{\partial}{\partial x_i}(q_k u_i p_i) = u_i \frac{\partial (p_i q_k)}{\partial x_i} + q_i p_k \frac{\partial u_k}{\partial x_i}. \tag{1.13c$'$}$$

We consider the vector $\boldsymbol{u} \equiv \boldsymbol{u}^{(1)}$ and the second-rank tensors $q^{(1,N)}$ and $p^{(1,N)}$ with the components u_i, $q_{kn}^{(1,N)}$, and $p_{jn}^{(1,N)}$, respectively. From (1.13a), (1.13b$'$), and (1.13c$'$), adding the missing subscript n to the functions q_k and p_j and assuming the summation over n from $-N$ to N, we can derive the necessary equalities:

$$\frac{\partial}{\partial x_k}(u_i q_{in}^{(1,N)} p_{kn}^{(1,N)}) = q_{in}^{(1,N)} p_{kn}^{(1,N)} \frac{\partial u_i}{\partial x_k} + u_i \frac{\partial (q_{in}^{(1,N)} p_{kn}^{(1,N)})}{\partial x_k}, \tag{1.14}$$

$$\frac{\partial}{\partial x_k}(u_k q_{in}^{(1,N)} p_{in}^{(1,N)}) = u_i \frac{\partial}{\partial x_k}(\delta_{ik} q_{jn}^{(1,N)} p_{jn}^{(1,N)}) + q_{in}^{(1,N)} p_{in}^{(1,N)} \frac{\partial u_k}{\partial x_k}, \tag{1.15}$$

$$\frac{\partial}{\partial x_k}(u_i p_{in}^{(1,N)} q_{kn}^{(1,N)}) = u_i \frac{\partial}{\partial x_i}(p_{in}^{(1,N)} q_{kn}^{(1,N)}) + q_{in}^{(1,N)} p_{kn}^{(1,N)} \frac{\partial u_k}{\partial x_i}. \tag{1.16}$$

It is probable that equalities (1.14)–(1.16) can be obtained as combinations of identities of the form (1.11) and (1.12) if we decompose the tensor-functions $q^{(1,N)}$ and $p^{(1,N)}$ into irreducible representations. We did not investigate this problem.

Identities (1.11), (1.12), and (1.14)–(1.16), which will be referred to as the rules of generalized vector calculus, will serve as the basis of our further constructions.

2. Construction of Complicated Systems. The simplest systems of Galilean-invariant thermodynamically consistent equations, whose properties were discussed in detail in [1], are written as follows:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L_{q_0})}{\partial x_k} = 0 \quad \text{(conservation of mass)},$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} = 0 \quad \text{(conservation of momentum)},$$

$$\frac{\partial L_{\boldsymbol{q}}}{\partial t} + \frac{\partial (u_k L_{\boldsymbol{q}})}{\partial x_k} = -\boldsymbol{\varphi}, \tag{2.1}$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial (u_k L_T)}{\partial x_k} = \frac{\boldsymbol{q}\boldsymbol{\varphi}}{T} \quad \text{(entropy equation)},$$

$$\frac{\partial E}{\partial t} + \frac{\partial [u_k (E + L)]}{\partial x_k} = 0 \quad \text{(conservation of energy)}.$$

In these equations, u_{-1}, u_0, and u_1 are the velocity components, T is the temperature, q_0 is a scalar variable, whereas $\boldsymbol{q}$ is a variable (as a rule, a vector variable) of the same dimension as the right side $\boldsymbol{\varphi}$, and $\boldsymbol{q}\boldsymbol{\varphi}$ is the scalar product of vector multipliers. If we denote the components of the vectors $\boldsymbol{q}$ and $\boldsymbol{\varphi}$ by q_γ and φ_γ, then the potential L that generates system (2.1) should be given in the form of an "equation of state":

$$L = L(q_0, u_{-1}, u_0, u_1, \boldsymbol{q}, T) \equiv L(q_0, u_{-1}, u_0, u_1, q_1, q_2, \ldots, T).$$

The Legendre transform of L is denoted by E:

$$E = q_0 L_{q_0} + u_{-1}L_{u_{-1}} + u_0 L_{u_0} + u_1 L_{u_1} + \sum_{\gamma\neq 0} q_\gamma L_{q_\gamma} + TL_T - L$$

$$\equiv q_0 L_{q_0} + u_k L_{u_k} + q_\gamma L_{q_\gamma} + TL_T - L \equiv q_0 L_{q_0} + \boldsymbol{q}L_{\boldsymbol{q}} + \boldsymbol{u}L_{\boldsymbol{u}} + TL_T - L.$$

The components of the vector $\boldsymbol{\varphi}$ in equations admissible from the viewpoint of thermodynamics should ensure the positiveness of the right side of the penultimate equation of system (2.1) (the law of increase of entropy).

The law of conservation of energy [the last equation in (2.1)] is a consequence of all the remaining equations of this system. In [1] it was shown that if the generating potential L remains invariant under rotations of the coordinate axes and under their associated transformations of the vector-functions appearing in (2.1), then system (2.1) is also invariant under rotations. It is also invariant under conversion to a coordinate system moving at constant velocity relative to the initial coordinate system if

$$L = \Lambda(q_0 + u_i u_i/2, \boldsymbol{q}, T). \tag{2.2}$$

In other words, under assumption (2.2) the system of equations (2.1) is Galilean-invariant.

We assume that the vector $\boldsymbol{q}$ is composed of scalar components, vector components $\boldsymbol{q}^{(A)}$, and tensor components $q^{(1,A)}$ with various weights A. Different components can have the same weight. We also note that under rotations the components of the vectors $L_{\boldsymbol{u}}$ and $L_{\boldsymbol{q}}$ are transformed by the same representations as the corresponding components of the vectors $\boldsymbol{u}$ and $\boldsymbol{q}$.

We can now proceed to construction of complicated equations.

The equations are constructed from parts. One such part is the simplest system (2.1) and the other parts are identities (1.11), (1.12), and (1.14)–(1.16). During construction, we introduce new terms that contain vector derivatives of unknown vector-functions into an equation of the initial simplest system. In doing so, we ensure that these terms are chosen from the aggregates entering in one or another identity and that, based on that identity, they ensure exact fulfilment of the law of conservation of mass (in the initial formulation) and the laws of conservation of momentum and energy. We always have to introduce additional terms into the energy fluxes, whereas additional terms in the momentum fluxes are sometimes introduced and sometimes are not.

Let the initial system consist of equalities whose "integrating multipliers" are q_0, u_i, $\boldsymbol{q}^{(A)}$, $\boldsymbol{r}^{(A)}$, and T, respectively:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial(u_k L)_{u_i}}{\partial x_k} = 0,$$

$$\frac{\partial L_{\boldsymbol{q}^{(A)}}}{\partial t} + \frac{\partial(u_k L_{\boldsymbol{q}^{(A)}})}{\partial x_k} = -\boldsymbol{\varphi}^{(A)}, \qquad \frac{\partial L_{\boldsymbol{r}^{(A)}}}{\partial t} + \frac{\partial(u_k L_{\boldsymbol{r}^{(A)}})}{\partial x_k} = -\boldsymbol{\psi}^{(A)}, \tag{2.3}$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{(\boldsymbol{q}^{(A)}, \boldsymbol{\varphi}^{(A)}) + (\boldsymbol{r}^{(A)}, \boldsymbol{\psi}^{(A)})}{T}.$$

Taking a linear combination of these equalities with the "integrating multipliers" as coefficients, we obtain an additional law of conservation of energy:

$$\frac{\partial E}{\partial t} + \frac{\partial[u_k(E + L)]}{\partial x_k} = 0,$$

where

$$E = q_0 L_{q_0} + (\boldsymbol{u}, L_{\boldsymbol{u}}) + (\boldsymbol{q}^{(A)}, L_{\boldsymbol{q}^{(A)}}) + (\boldsymbol{r}^{(A)}, L_{\boldsymbol{r}^{(A)}}) + TL_T - L.$$

From this initial system we construct a complicated system which differs from the initial one in additional terms that are introduced into the equations containing the derivatives of $L_{\boldsymbol{q}^{(A)}}$ and $L_{\boldsymbol{r}^{(A)}}$ with respect to t. Then, the modified equations take the form

$$\frac{\partial L_{\boldsymbol{q}^{(A)}}}{\partial t}+\frac{\partial(u_k L_{\boldsymbol{q}^{(A)}})}{\partial x_k}+\frac{\delta\sqrt{3}}{\sqrt{2A+1}}[\nabla\times\boldsymbol{r}^{(A)}]^{(A)}=-\boldsymbol{\varphi}^{(A)},$$
$$\frac{\partial L_{\boldsymbol{r}^{(A)}}}{\partial t}+\frac{\partial(u_k L_{\boldsymbol{r}^{(A)}})}{\partial x_k}-\frac{\delta\sqrt{3}}{\sqrt{2A+1}}[\nabla\times\boldsymbol{q}^{(A)}]^{(A)}=-\boldsymbol{\psi}^{(A)}. \tag{2.4}$$

Here δ is an arbitrary constant multiplier. Obviously, the left sides of the modified equations retain the divergent form.

If we repeat the derivation of the energy equation for the modified system described above, using a linear combination of equations with the same "integrating multipliers," then we obtain the equality

$$\frac{\partial E}{\partial t}+\frac{\partial\{u_k(E+L)+\delta[\boldsymbol{q}^{(A)}\times\boldsymbol{r}^{(A)}]_k^{(1)}\}}{\partial x_k}=0, \tag{2.5}$$

which differs from the corresponding initial equality in having the components $u_k(E+L)$ in the energy flux vector replaced by $u_k(E+L)+\delta[\boldsymbol{q}^{(A)}\times\boldsymbol{r}^{(A)}]_k^{(1)}$. In derivation of (2.5) we used identity (1.11).

By virtue of the invariance under rotations of the additionally introduced terms, the modified system remains rotationally invariant. The special dependence of the generating potential (2.2) on the velocity components u_i and the fact that the first equation of system (2.3) (the law of conservation of mass) is not changed in this modification ensure the invariance of the modified equations in conversion to a moving coordinate system that moves at constant velocity relative to the initial coordinate system ($x_k \to x_k - tU_k$ and $u_k \to u_k - U_k$). Here we do not dwell on elementary verification of the above statement which, in essence, is not different from the verification made in [1] for a simpler case. Here we need to use the fact that the new terms introduced in (2.4) are independent of u_k.

Thus, the modification of the simplest equations described here does not lead to violation of their Galilean invariance.

We consider another similar construction of complication of the initial system of equations whose unknowns include the vector-functions $\boldsymbol{q}^{(A)}$ and $\boldsymbol{r}^{(A+1)}$ transformed by representations of integer weights A and $A+1$ that differ by 1:

$$\frac{\partial L_{q_0}}{\partial t}+\frac{\partial(u_k L_{q_0})}{\partial x_k}=0,\qquad \frac{\partial L_{u_i}}{\partial t}+\frac{\partial(u_k L)_{u_i}}{\partial x_k}=0,$$
$$\frac{\partial L_{\boldsymbol{q}^{(A)}}}{\partial t}+\frac{\partial(u_k L_{\boldsymbol{q}^{(A)}})}{\partial x_k}=-\boldsymbol{\varphi}^{(A)},\qquad \frac{\partial L_{\boldsymbol{r}^{(A+1)}}}{\partial t}+\frac{\partial(u_k L_{\boldsymbol{r}^{(A+1)}})}{\partial x_k}=-\boldsymbol{\psi}^{(A+1)},$$
$$\frac{\partial L_T}{\partial t}+\frac{\partial(u_k L_T)}{\partial x_k}=\frac{(\boldsymbol{q}^{(A)},\boldsymbol{\varphi}^{(A)})+(\boldsymbol{r}^{(A+1)},\boldsymbol{\psi}^{(A+1)})}{T},\qquad \frac{\partial E}{\partial t}+\frac{\partial[u_k(E+L)]}{\partial x_k}=0.$$

An additional law of conservation of energy obtained as a linear combination of the first five equations with the coefficients q_0, u_i, $\boldsymbol{q}^{(A)}$, $\boldsymbol{r}^{(A+1)}$, and T is included in the system. Complication of this initial system will again be made with the use of identity (1.11). Introducing additional terms into the equations containing the derivatives of $L_{\boldsymbol{q}^{(A)}}$ and $L_{\boldsymbol{r}^{(A+1)}}$ with respect to time t, we write them as follows:

$$\frac{\partial L_{\boldsymbol{q}^{(A)}}}{\partial t}+\frac{\partial(u_k L_{\boldsymbol{q}^{(A)}})}{\partial x_k}+\frac{\delta\sqrt{3}}{\sqrt{2A+1}}[\nabla\times\boldsymbol{r}^{(A+1)}]^{(A)}=-\boldsymbol{\varphi}^{(A)},$$
$$\frac{\partial L_{\boldsymbol{r}^{(A+1)}}}{\partial t}+\frac{\partial(u_k L_{\boldsymbol{r}^{(A+1)}})}{\partial x_k}+\frac{\delta\sqrt{3}}{\sqrt{2A+3}}[\nabla\times\boldsymbol{q}^{(A)}]^{(A+1)}=-\boldsymbol{\psi}^{(A+1)}. \tag{2.6}$$

The above modification of these two vector equations leads to modification of the energy equation:

$$\frac{\partial E}{\partial t}+\frac{\partial\{u_k(E+L)+\delta[\boldsymbol{q}^{(A)}\times\boldsymbol{r}^{(A+1)}]_k^{(1)}\}}{\partial x_k}=0.$$

In the version of complication considered above, the Galilean invariance of the complicated system is proved by verbal repetition of the same arguments as in the previous example. We note that in the versions of complication of the initial system described here, no corrections were made in the equations modeling the laws of conservation of mass and momentum.

The law of conservation of mass (the first of the equalities which appear in the modified system) is subjected to correction only if the scalar q_0 is chosen as $\boldsymbol{q}^{(A)}$ in the second version of modification. We abandon such modifications since the additional terms in the mass fluxes that arise from correction contradict the proof of the invariance of the equations in conversion to a moving coordinate system (see [1, Sec. 1]).

In complications based on versions of identity (1.12), corrections appear in the momentum equations. We give an example of complication based on a version of identity (1.12) with $L = M = A$ and $K = 1$ [in the notation of the velocity $\boldsymbol{u}$, we omit the symbol that indicates the weight (1) of the representation]:

$$\frac{\partial}{\partial x_k}\,[[\boldsymbol{q}^{(A)} \times L_{\boldsymbol{q}^{(A)}}]^{(1)} \times \boldsymbol{u}]^{(1)}_k = (\boldsymbol{u}, [\nabla \times [\boldsymbol{q}^{(A)} \times L_{\boldsymbol{q}^{(A)}}]^{(1)}]^{(1)}) - \sqrt{3/(2A+1)}\,(\boldsymbol{q}^{(A)}, [L_{\boldsymbol{q}^{(A)}} \times [\nabla \times \boldsymbol{u}]^{(1)}]^{(A)}).$$

We now give a complicated system that includes the equation of conservation of mass

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0,$$

the modified momentum equation

$$\frac{\partial L_{\boldsymbol{u}}}{\partial t} + \frac{\partial}{\partial x_k}\Big\{(u_k L)_{\boldsymbol{u}} + \frac{\delta}{\sqrt{3}}[\nabla \times [\boldsymbol{q}^{(A)} \times L_{\boldsymbol{q}^{(A)}}]^{(1)}]^{(1)}\Big\} = 0,$$

the vector equation

$$\frac{\partial L_{\boldsymbol{q}^{(A)}}}{\partial t} + \frac{\partial(u_k L_{\boldsymbol{q}^{(A)}})}{\partial x_k} - \frac{\delta\sqrt{3}}{\sqrt{2A+1}}\,[L_{\boldsymbol{q}^{(A)}} \times [\nabla \times \boldsymbol{u}]^{(1)}]^{(A)} = -\boldsymbol{\varphi}^{(A)},$$

which has lost the form of a conservation law since its left side is supplemented by a nondivergent term, whose each component is a linear combination of the first derivatives of the velocity components, the compensating equation for entropy

$$\frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{(\boldsymbol{q}^{(A)}, \boldsymbol{\varphi}^{(A)})}{T},$$

and the equality

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_k}\{(E+L)u_k + \delta[[\boldsymbol{q}^{(A)} \times L_{\boldsymbol{q}^{(A)}}]^{(1)} \times \boldsymbol{u}]^{(1)}_k\} = 0.$$

In the above list of equations, the last equation (the law of conservation of energy) is a consequence of the preceding equations. The invariance of the equations under rotations of the coordinate system is beyond doubt. The invariance under conversion to a moving (at constant velocity) coordinate system follows from the fact that in making the complication we do not change the first equation (the law of conservation of mass), supplementing the remaining equations (except for the last equation, which is a consequence of the preceding ones) with new terms that are independent of the components u_j or depend only on the derivatives $\partial u_j/\partial x_i$. These derivatives do not change when u_j is replaced by $u_j + U_j$ ($U_j = \text{const}$). We recall (see [1, Sec. 1]) that in the proof of invariance, we have to add the equation of conservation of mass (which is not modified) multiplied by constant coefficients to the momentum equations. Naturally, the invariance is only due to the fact that the generating potential (2.2) is given by a function that is invariant under rotations. After similar transformations, other versions of identities united by equality (1.5) lead to new versions of Galilean-invariant complicated systems. We shall not dwell on this now.

We now turn to modifications which are based on identities (1.14)–(1.16). In this case, it is natural to use the generalized tensor notation described in Sec. 1.

We use the system

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial(u_k L)_{u_i}}{\partial x_k} = 0,$$

$$\frac{\partial L_{q^{(1,A)}_{ja}}}{\partial t} + \frac{\partial(u_k L_{q^{(1,A)}_{ja}})}{\partial x_k} = -\varphi^{(1,A)}_{ja}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q^{(1,A)}_{ja}\varphi^{(1,A)}_{ja}}{T}, \tag{2.7}$$

$$\frac{\partial E}{\partial t} + \frac{\partial[u_k(E+L)]}{\partial x_k} = 0$$

as the initial system and introduce additional terms into the left sides of some equations.

We assume that the result of the modification is as follows:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial[(u_k L)_{u_i} - q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t} + \frac{\partial(u_k L_{q_{ja}^{(1,A)}})}{\partial x_k} - L_{q_{ka}^{(1,A)}} \frac{\partial u_j}{\partial x_k} = -\varphi_{ja}^{(1,A)}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q_{ja}^{(1,A)} \varphi_{ja}^{(1,A)}}{T}, \tag{2.8}$$

$$\frac{\partial E}{\partial t} + \frac{\partial[u_k(E+L) - u_i q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0.$$

Just as in the initial system (2.7), the last equation in the modified system is a linear combination of the preceding equations with the coefficients q_0, u_i, $q_{ja}^{(1,A)}$, and T, respectively. To justify this statement, we need to use identity (1.14) after making the following change of notation in that identity:

$$N \to A, \quad n \to a, \quad p_{kn}^{(1,N)} \to L_{q_{ka}^{(1,A)}}, \quad q_{in}^{(1,N)} \to q_{ia}^{(1,A)}.$$

We note that the equations on the left of the second line of system (2.8) have lost the divergent form as a result of modification and are no longer conservation laws. In contrast, the laws of conservation of mass, momentum, and energy are not violated, although the last two laws have changed.

Similarly, modified equations differing from (2.8) only in certain signs are constructed from the initial system (2.7) using identity (1.14):

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial[(u_k L)_{u_i} + q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t} + \frac{\partial(u_k L_{q_{ja}^{(1,A)}})}{\partial x_k} + L_{q_{ka}^{(1,A)}} \frac{\partial u_j}{\partial x_k} = -\varphi_{ja}^{(1,A)}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q_{ja}^{(1,A)} \varphi_{ja}^{(1,A)}}{T}, \tag{2.9}$$

$$\frac{\partial E}{\partial t} + \frac{\partial[u_k(E+L) + u_i q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0;$$

identity (1.16) leads to another possible modification:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial[(u_k L)_{u_i} - \delta_{ik} q_{ja}^{(1,A)} L_{q_{ja}^{(1,A)}}]}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t} + \frac{\partial(u_k L_{q_{ja}^{(1,A)}})}{\partial x_k} - L_{q_{ja}^{(1,A)}} \frac{\partial u_k}{\partial x_k} = -\varphi_{ja}^{(1,A)}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q_{ja}^{(1,A)} \varphi_{ja}^{(1,A)}}{T}, \tag{2.10}$$

$$\frac{\partial E}{\partial t} + \frac{\partial[u_k(E+L - q_{ia}^{(1,A)} L_{q_{ia}^{(1,A)}})]}{\partial x_k} = 0.$$

We note that in constructing modifications containing the laws of conservation of mass, momentum, and energy, we can use several identities (1.14)–(1.16) simultaneously. We give a possible example of a modification constructed in this way:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial[(u_k L)_{u_i} - \delta_{ik} q_{ja}^{(1,A)} L_{q_{ja}^{(1,A)}} + q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t} + \frac{\partial(u_k L_{q_{ja}^{(1,A)}})}{\partial x_k} - L_{q_{ja}^{(1,A)}} \frac{\partial u_k}{\partial x_k} + L_{q_{ka}^{(1,A)}} \frac{\partial u_k}{\partial x_j} = -\varphi_{ja}^{(1,A)}, \tag{2.11}$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q_{ja}^{(1,A)} \varphi_{ja}^{(1,A)}}{T}, \qquad \frac{\partial E}{\partial t} + \frac{\partial[u_k(E+L - q_{ja}^{(1,A)} L_{q_{ja}^{(1,A)}}) + u_i q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0.$$

The left sides of the above equations contain additional terms (relative to the initial system), which were used earlier in constructing both Eqs. (2.9) and system (2.10).

Additional [to system (2.7)] terms of the modified systems (2.8)–(2.11) do not violate Galilean invariance since the modification does not touch upon the first equation (the law of conservation of mass) and in all equations, except for the last one, it uses only the derivatives $\partial u_i/\partial x_k$, so that the additional terms are not changed when constants are added to u_i. In addition, it is easy to check that the additional terms are invariant under rotations. The last equality in (2.11) is a consequence of all the preceding equalities.

We recall that we use the generating potential L of the special form (2.2).

3. Modifications Equivalent to Symmetric Hyperbolic Systems. If the generating potential is a convex function of its arguments, then simple thermodynamically consistent systems of conservation laws can always be written in the form of a symmetric hyperbolic system. Although this fact is generally known at present, we need to dwell on it because this fact is used in investigation of the modified equations.

If a simple system does not include the last (energy) equation, which is a consequence of the remaining equations, then this system can be written as follows $[L = L(r_1, r_2, \ldots, r_N),\ M^{(k)} = u_k L = M^{(k)}(r_1, r_2, \ldots, r_N)]$:

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial M^{(k)}_{r_i}}{\partial x_k} = f_i \qquad (3.1)$$

(all unknowns here are renamed as r_i). Equations (3.1) can be rewritten in quasilinear form

$$L_{r_i r_j}\frac{\partial r_j}{\partial t} + M^{(k)}_{r_i r_j}\frac{\partial r_j}{\partial x_k} = f_i, \qquad (3.2)$$

so that the matrices of the coefficients at the derivatives turn out to be symmetric. The positive definiteness of the matrix $L_{r_i r_j}$ follows from the assumption that the generating potential L is convex.

Describing the procedure of constructing modified equations in Sec. 2, we singled out constructions in which additional terms are not introduced into the equalities that describe the law of conservation of momentum. It turns out that in this case the system constructed can also be written in the form (3.1) but with modified potentials $M^{(k)}$, and therefore, the justification of hyperbolicity based on consideration of its quasilinear version (3.2) remains valid.

We show the validity of this statement using as an example Eqs. (2.4) and (2.6), which were considered earlier.

Equations (2.4) are written in a more detailed form using the Clebsch–Gordan coefficients:

$$\frac{\partial L_{q^{(A)}_\alpha}}{\partial t} + \frac{\partial (u_k L_{q^{(A)}_\alpha})}{\partial x_k} + \delta\sqrt{\frac{3}{2A+1}}\, G^{\alpha[k,b]}_{A[1,A]}\frac{\partial r^{(A)}_b}{\partial x_k} = -\varphi^{(A)}_\alpha,$$

$$\frac{\partial L_{r^{(A)}_\beta}}{\partial t} + \frac{\partial (u_k L_{r^{(A)}_\beta})}{\partial x_k} - \delta\sqrt{\frac{3}{2A+1}}\, G^{\beta[k,a]}_{A[1,A]}\frac{\partial q^{(A)}_a}{\partial x_k} = -\psi^{(A)}_\beta.$$

Using the properties of the Clebsch–Gordan coefficients

$$G^{\alpha[k,b]}_{A[1,A]} = -G^{b[k,\alpha]}_{A[1,A]} = \sqrt{(2A+1)/3}\, G^{k[b,\alpha]}_{1[A,A]} = -\sqrt{(2A+1)/3}\, G^{k[\alpha,b]}_{1[A,A]}$$

and setting

$$F^{(k)} = \delta(G^{k}_{1[A,A]}\boldsymbol{r}^{(A)}, \boldsymbol{q}^{(A)}) \equiv \delta G^{k[b,\alpha]}_{1[A,A]} r^{(A)}_b q^{(A)}_\alpha,$$

we can establish that if we modify a simple system as is indicated in (2.4), then in Eqs. (3.1) we need to use the modified potentials $\hat{M}^{(k)} = M^{(k)} + F^{(k)}$ instead of the potentials $M^{(k)}$. Therefore, in the quasilinear formulation (3.2), the elements $M^{(k)}_{r_i r_j}$ are replaced by $\hat{M}^{(k)}_{r_i r_j}$, and it is evident that in this case the coefficient matrices remain symmetric. The matrix of coefficients at the derivatives with respect to t does not change at all, thus remaining symmetric and positive definite. Consequently, the modification considered preserves the symmetric hyperbolicity of the equations. Eqs. (2.6) can be written in greater detail:

$$\frac{\partial L_{q^{(A)}_\alpha}}{\partial t} + \frac{\partial (u_k L_{q^{(A)}_\alpha})}{\partial x_k} + \delta\sqrt{\frac{3}{2A+1}}\, G^{\alpha[k,a]}_{A[1,A+1]}\frac{\partial r^{(A+1)}_a}{\partial x_k} = -\varphi^{(A)}_\alpha,$$

$$\frac{\partial L_{r^{(A+1)}_\beta}}{\partial t} + \frac{\partial (u_k L_{r^{(A+1)}_\beta})}{\partial x_k} + \delta\sqrt{\frac{3}{2A+3}}\, G^{\beta[k,b]}_{A+1[1,A]}\frac{\partial q^{(A)}_b}{\partial x_k} = -\psi^{(A+1)}_\beta.$$

Again using the properties of the Clebsch–Gordan coefficients

$$\sqrt{3/(2A+1)}\,G^{\alpha[k,a]}_{A[1,A+1]} = \sqrt{3/(2A+1)}G^{a[k,\alpha]}_{A+1[1,A]} = G^{k[a,\alpha]}_{1[A+1,A]} = G^{k[\alpha,a]}_{1[A,A+1]}$$

and setting

$$F^{(k)} = \delta G^{k[a,\alpha]}_{1[A+1,A]} q^{(A)}_a r^{(A+1)}_\alpha = \delta(G^k_{1[A+1,A]}\boldsymbol{q}^{(A)}, \boldsymbol{r}^{(A+1)}),$$

we see that the modification described is again reduced to replacement of the potentials $M^{(k)}$ by the corresponding potentials $\hat{M}^{(k)} = M^{(k)} + F^{(k)}$, which, as was already noted, does not violate the symmetric hyperbolicity of the equations.

However, it is not clear at all whether the constructions given at the end of Sec. 2, which include new terms in the momentum flux, lead to hyperbolic modified equations. It is most probable that, in general, it is not so. At the same time, hyperbolic modifications exist among systems with modified momentum, too. We show that modifications of systems (2.8) and (2.11) can be written in the form of symmetric hyperbolic systems. This reduction is based on the fact that Eqs. (2.8) and (2.11) have characteristics $dx_i/dt = u_i$ — streamlines. The relations along these characteristics can be found and used to symmetrize the coefficient matrices of the quasilinear form of the above-mentioned equations.

We begin with construction of the relations along the streamlines for the equations of system (2.8). For this, it suffices to use the following equalities from that system:

$$\frac{\partial L_{q^{(1,A)}_{ja}}}{\partial t} + \frac{\partial(u_k L_{q^{(1,A)}_{ja}})}{\partial x_k} - L_{q^{(1,A)}_{ka}}\frac{\partial u_j}{\partial x_k} = -\varphi^{(1,A)}_{ja}.$$

Differentiating each of these equalities with respect to x_j and summing over j, we obtain the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L_{q^{(1,A)}_{ja}}}{\partial x_j}\right) + \frac{\partial}{\partial x_k}\left(u_k\,\frac{\partial L_{q^{(1,A)}_{ja}}}{\partial x_j}\right) = -\frac{\partial \varphi^{(1,A)}_{ja}}{\partial x_j}. \tag{3.3}$$

Comparing this equation with the law of conservation of mass

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0$$

and introducing the notation

$$D^{(A)}_a = \frac{1}{L_{q_0}}\frac{\partial}{\partial x_j} L_{q^{(1,A)}_{ja}}, \tag{3.4}$$

we see that

$$\frac{\partial D^{(A)}_a}{\partial t} + u_k\frac{\partial D^{(A)}_a}{\partial x_k} = -\frac{1}{L_{q_0}}\frac{\partial\varphi^{(1,A)}_{ja}}{\partial x_j}. \tag{3.5}$$

Using the notation (3.4) and differentiating, we transform the law of conservation of momentum

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial[(u_k L)_{u_i} - q^{(1,A)}_{ia} L_{q^{(1,A)}_{ka}}]}{\partial x_k} = 0$$

into

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial(u_k L)_{u_i}}{\partial x_k} - L_{q^{(1,A)}_{ka}}\frac{\partial q^{(1,A)}_{ia}}{\partial x_k} = q^{(1,A)}_{ia}\frac{\partial L_{q^{(1,A)}_{ka}}}{\partial x_k} \equiv q^{(1,A)}_{ia} L_{q_0} D^{(A)}_a.$$

After that, system (2.8) supplemented with equality (3.5) becomes:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial(u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial(u_k L)_{u_i}}{\partial x_k} - L_{q^{(1,A)}_{ka}}\frac{\partial q^{(1,A)}_{ia}}{\partial x_k} = L_{q_0} q^{(1,A)}_{ia} D^{(A)}_a,$$

$$\frac{\partial L_{q^{(1,A)}_{ia}}}{\partial t} + \frac{\partial(u_k L_{q^{(1,A)}_{ia}})}{\partial x_k} - L_{q^{(1,A)}_{ka}}\frac{\partial u_i}{\partial x_k} = -\varphi^{(1,A)}_{ia}, \tag{3.6}$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial(u_k L_T)}{\partial x_k} = \frac{q^{(1,A)}_{ja}\varphi^{(1,A)}_{ja}}{T}, \qquad \frac{\partial D^{(A)}_a}{\partial t} + u_k\frac{\partial D^{(A)}_a}{\partial x_k} = -\frac{1}{L_{q_0}}\frac{\partial \varphi^{(1,A)}_{ja}}{\partial x_j}.$$

This system differs from the simplest initial Galilean-invariant and thermodynamically consistent equations only in the presence of the last equation and additional terms (the last terms) in some equations. Reducing the simplest equations to a quasilinear form with symmetric coefficient matrices , we can easily see that this symmetry is preserved after inclusion of the above-mentioned additional terms in the equations. The matrices at the derivatives with respect to t do not change. This implies the symmetric hyperbolicity of Eqs. (3.6) but only if L is a convex function of its arguments. It is easy to check that inclusion of the last equality in system (3.6) does not violate its symmetric hyperbolicity as well.

The Cauchy problem for system (3.6) is uniquely solvable for all sufficiently smooth initial data, but equality (3.4) may fail on its solutions. But if the initial data satisfy equality (3.4), then the uniqueness of the solution and a comparison of Eqs. (3.3) and (3.5) imply that (3.4) is fulfilled identically everywhere. Thus, the symmetric hyperbolicity of one of the equivalent versions (3.6) of Eqs. (2.8) is justified.

Assuming that at the initial time we have $D_a^{(A)} = 0$ and $\varphi^{(1,A)} = 0$, we come to the following compatible overdetermined system of conservation laws:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L_{u_i} + \delta_{ik} L - q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}})}{\partial x_k} = 0,$$

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t} + \frac{\partial (u_k L_{q_{ja}^{(1,A)}} - u_j L_{q_{ka}^{(1,A)}})}{\partial x_k} = 0, \qquad \frac{\partial}{\partial x_k} L_{q_{ka}^{(1,A)}} = 0, \tag{3.7}$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial (u_k L_T)}{\partial x_k} = 0, \qquad \frac{\partial E}{\partial t} + \frac{\partial [u_k (E + L) - u_i q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}}]}{\partial x_k} = 0.$$

We recall that $E = q_0 L_{q_0} + u_i L_{u_i} + q_{ja}^{(1,A)} L_{q_{ja}^{(1,A)}} + T L_T - L$, $L = \Lambda(q_0 + u_i u_i/2, q^{(1,A)}, T)$, and L is assumed to be a convex function of its arguments.

A similar derivation of the relations along the streamlines for Eqs. (2.11) with zero right sides will be described briefly. We use the equations of system (2.11)

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t} + \frac{\partial (u_k L_{q_{ja}^{(1,A)}})}{\partial x_k} - L_{q_{ja}^{(1,A)}} \frac{\partial u_k}{\partial x_k} + L_{q_{ka}^{(1,A)}} \frac{\partial u_k}{\partial x_j} = 0. \tag{3.8}$$

Differentiating equalities (3.8) and introducing the notation

$$\frac{\partial L_{q_{ka}^{(1,A)}}}{\partial x_m} - \frac{\partial L_{q_{ma}^{(1,A)}}}{\partial x_k} = -e_{kmj} \omega_{ja}^{(1,A)},$$

we can justify the equality

$$\frac{\partial \omega_{ja}^{(1,A)}}{\partial t} + \frac{\partial (u_k \omega_{ja}^{(1,A)} - u_j \omega_{ka}^{(1,A)})}{\partial x_k} = 0, \tag{3.9}$$

which is an analog of equality (3.3) used in the above analysis of Eqs. (2.8). In particular, it follows from (3.9) that if the right sides are zeros and all $\omega_{ja}^{(1,A)}$ are equal to zero at the initial time, then this equality will be preserved subsequently. Reduction of Eqs. (2.11) with zero right sides (!!!) to the form of a symmetric hyperbolic system is carried out under the assumption that $\omega_{ja}^{(1,A)} = 0$. This equality makes it possible to transform the equations of the law of conservation of momentum

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L_{u_i} + \delta_{ik} L - \delta_{ik} q_{ja}^{(1,A)} L_{q_{ja}^{(1,A)}} + q_{ia}^{(1,A)} L_{q_{ka}^{(1,A)}})}{\partial x_k} = 0$$

to a nondivergent but more convenient form:

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} + L_{q_{ia}^{(1,A)}} \frac{\partial q_{ka}^{(1,A)}}{\partial x_k} - L_{q_{ka}^{(1,A)}} \frac{\partial q_{ia}^{(1,A)}}{\partial x_k} = 0.$$

As a result, we arrive at the following equations equivalent (under above the assumptions) to system (2.11):

$$\frac{\partial L_{q_0}}{\partial t}+\frac{\partial(u_kL_{q_0})}{\partial x_k}=0,\qquad \frac{\partial L_{u_i}}{\partial t}+\frac{\partial(u_kL)_{u_i}}{\partial x_k}+L_{q_{ia}^{(1,A)}}\frac{\partial q_{ia}^{(1,A)}}{\partial x_k}-L_{q_{ka}^{(1,A)}}\frac{\partial q_{ka}^{(1,A)}}{\partial x_i}=0,$$
$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t}+\frac{\partial(u_kL_{q_{ja}^{(1,A)}})}{\partial x_k}+L_{q_{ka}^{(1,A)}}\frac{\partial u_k}{\partial x_k}-L_{q_{ja}^{(1,A)}}\frac{\partial u_k}{\partial x_j}=0,\qquad \frac{\partial L_T}{\partial t}+\frac{\partial(u_kL_T)}{\partial x_k}=0. \tag{3.10}$$

Equalities (3.10) are now easily reduced to a quasilinear form with symmetric matrices of coefficients at the derivatives with respect to t and x_k, whence symmetric hyperbolicity follows in the case of a convex generating function L.

In conclusion, we give an overdetermined system of conservation laws which is similar to (3.7) and is equivalent to Eqs. (2.11) supplemented with one more equation for the unknown n:

$$\frac{\partial L_{q_0}}{\partial t}+\frac{\partial(u_kL_{q_0})}{\partial x_k}=0,\qquad \frac{\partial L_{u_i}}{\partial t}+\frac{\partial[u_kL_{u_i}+\delta_{ik}(L-q_{ja}^{(1,A)}L_{q_{ja}^{(1,A)}})+q_{ia}^{(1,A)}L_{q_{ka}^{(1,A)}}]}{\partial x_k}=0,$$

$$\frac{\partial L_{q_{ja}^{(1,A)}}}{\partial t}+\frac{\partial(u_lL_{q_{la}^{(1,A)}})}{\partial x_j}=-e_{jlm}u_l\omega_{ma}^{(1,A)},\qquad \frac{\partial L_{q_{ka}^{(1,A)}}}{\partial x_m}-\frac{\partial L_{q_{ma}^{(1,A)}}}{\partial x_k}=-e_{kmj}\omega_{ja}^{(1,A)},$$

$$\frac{\partial \omega_{ja}^{(1,A)}}{\partial t}+\frac{\partial(u_k\omega_{ja}^{(1,A)}-u_j\omega_{ka}^{(1,A)})}{\partial x_k}=0,\qquad \frac{\partial L_n}{\partial t}+\frac{\partial(u_kL_n)}{\partial x_k}=-\nu,$$

$$\frac{\partial L_T}{\partial t}+\frac{\partial(u_kL_T)}{\partial x_k}=\frac{n\nu}{T},\qquad \frac{\partial E}{\partial t}+\frac{\partial[u_k(E+L-q_{ja}^{(1,A)}L_{q_{ja}^{(1,A)}})+u_iq_{ia}^{(1,A)}L_{q_{ka}^{(1,A)}}]}{\partial x_k}=0.$$

Specific examples from mathematical physics containing equations constructed from the elements described in this and previous sections will be given in Sec. 4.

4. Two Concrete Examples. In this section we shall show how rather complex equations of mathematical physics can be obtained from the simplest Galilean-invariant thermodynamically consistent conservation laws using the modifications described in Secs. 2 and 3. The present study was carried out with the aim to systematize the widest possible class of such equations and is a continuation of a series of studies on standardization of formulas contained in various monographs on physics such as [6–8] and in a number of papers. The results of this work are described in papers [2–4], which give equations written in generating potentials. In the present work, we confine ourselves to generating potentials of special form and describe in detail the "constructions" of equations using those potentials.

In Sec. 2, a modified system (2.11) is described. We dwell on the version that differs from (2.11) by specification of the tensor variables $q_{ja}^{(1,A)}$; namely, we set $q_{i0}^{(1,0)}=j_i$ $(i=0,\pm1)$. Then, the system is written as

$$\frac{\partial L_{q_0}}{\partial t}+\frac{\partial(u_kL_{q_0})}{\partial x_k}=0,\qquad \frac{\partial L_{u_i}}{\partial t}+\frac{\partial[(u_kL)_{u_i}-\delta_{ik}j_mL_{j_m}+j_iL_{j_k}]}{\partial x_k}=0,$$
$$\frac{\partial L_{j_i}}{\partial t}+\frac{\partial(u_kL_{j_i})}{\partial x_k}-L_{j_i}\frac{\partial u_k}{\partial x_k}+L_{j_k}\frac{\partial u_k}{\partial x_i}=0,\qquad \frac{\partial L_T}{\partial t}+\frac{\partial(u_kL_T)}{\partial x_k}=\frac{\nu n}{T}, \tag{4.1}$$
$$\frac{\partial L_n}{\partial t}+\frac{\partial(u_kL_n)}{\partial x_k}=-\nu,\qquad \frac{\partial E}{\partial t}+\frac{\partial[u_k(E+L-j_mL_{j_m})+u_mj_mL_{j_k}]}{\partial x_k}=0.$$

We also assume that

$$L=\Lambda(q_0+u_iu_i/2,j_{-1},j_0,j_1,n,T),\quad E=q_0L_{q_0}+u_kL_{u_k}+j_mL_{j_m}+nL_n+TL_T-L. \tag{4.2}$$

It was shown in Sec. 3 that Eqs. (2.11) (with zero right sides) can be reduced to a symmetric hyperbolic system. This reduction is based on additional relations that follow from the equations and in the version (4.2) have the form

$$\frac{\partial \omega_r}{\partial t}+\frac{\partial(u_k\omega_r-u_r\omega_k)}{\partial x_k}=0, \tag{4.3}$$

$$\omega_{-1}=\frac{\partial L_{j_1}}{\partial x_0}-\frac{\partial L_{j_0}}{\partial x_1}=0,\qquad \omega_0=\frac{\partial L_{j_{-1}}}{\partial x_1}-\frac{\partial L_{j_1}}{\partial x_{-1}}=0,\qquad \omega_1=\frac{\partial L_{j_0}}{\partial x_{-1}}-\frac{\partial L_{j_{-1}}}{\partial x_0}=0. \tag{4.4}$$

Using new variables ω_m, we can rewrite equations of the third group from system (4.1) in the form

$$\frac{\partial L_{j_i}}{\partial t} + \frac{\partial (u_m L_{j_m})}{\partial x_i} - e_{ilr} u_l \omega_r = 0. \tag{4.5}$$

We recall that exactly these equations were used to derive relations (4.3).

We now subject Eqs. (4.5) and the penultimate equation of system (4.1) to one more modification, which, in essence, coincides with the simplest version of modification of (2.7) from Sec. 2. We give its result:

$$\frac{\partial L_n}{\partial t} + \frac{\partial (u_k L_n + j_k)}{\partial x_k} = -\nu, \qquad \frac{\partial L_{j_i}}{\partial t} + \frac{\partial (u_m L_{j_m} + n)}{\partial x_k} = 0. \tag{4.6}$$

We note that the additional terms n and j_k, which, during modification, were brought under the symbol of the derivatives $\partial/\partial x_k$, do not violate the validity of Eqs. (4.3), which follow from (4.5) as well as from (4.6).

At the same time, as was noted in Sec. 2, the modification performed leads to the appearance of additional terms in the energy fluxes. The energy equation becomes

$$\frac{\partial E}{\partial t} + \frac{\partial [u_k (E + L - j_m L_{j_m}) + u_m j_m L_{j_k} + n j_k]}{\partial x_k} = 0.$$

Thus, as a result of the modifications described above, we come to the following overdetermined but compatible system:

$$\begin{gathered}
\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial [(u_k L)_{u_i} - \delta_{ik} j_m L_{j_m} + j_i L_{j_k}]}{\partial x_k} = 0, \\
\frac{\partial L_{j_i}}{\partial t} + \frac{\partial (u_m L_{j_m} + n)}{\partial x_k} = 0, \qquad \frac{\partial L_{j_k}}{\partial x_m} - \frac{\partial L_{j_m}}{\partial x_k} = 0, \\
\frac{\partial L_n}{\partial t} + \frac{\partial (u_k L_n + j_k)}{\partial x_k} = -\nu, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial (u_k L_T)}{\partial x_k} = \frac{n\nu}{T}, \\
\frac{\partial E}{\partial t} + \frac{\partial [u_k (E + L - j_m L_{j_m}) + u_m j_m L_{j_k} + n j_k]}{\partial x_k} = 0.
\end{gathered} \tag{4.7}$$

The first two equations in system (4.7) model the laws of conservation of mass and momentum. The last but one equation is the compensating equation for entropy, and the equation which precedes it is the equation for the "chemical potential" n, whose gradient gives rise to superfluid flux j. The last equation in the system is the law of conservation of energy.

System (4.7) is close to the formalization of the equations for superfluid helium described in [8] and schematized in [4]. An additional equation for the "chemical potential" n is included in Eqs. (4.7), whereas in [8] its role was played by q_0. Unfortunately, in this case, the superfluid flux j enters in the first equation (the law of conservation of mass), and one of the postulates underlying our investigation is thus violated. The justification of the Galilean invariance in our work is based on this postulate.

We now turn to the compatible overdetermined system (3.7), which was constructed in Sec. 3 with the help of one modification considered there. We again confine ourselves to the simplest version, setting $A = 0$ and using the compact notation b_k for the variables $q_{k0}^{(1,0)}$. At the same time, instead of system (3.7) we consider its simple generalization to the case where the medium is characterized by two temperatures and terms describing viscous dissipation and heat conduction are included in the momentum flux and in the energy equation:

$$\begin{gathered}
\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L_{q_0})}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial}{\partial x_k}\Big(u_k L_{u_i} + \delta_{ik} L - b_i L_{b_k} + \mu \frac{\partial u_i}{\partial x_k}\Big) = 0, \\
\frac{\partial L_{b_i}}{\partial t} + \frac{\partial (u_k L_{b_i} - u_i L_{b_k})}{\partial x_k} = 0, \qquad \frac{\partial}{\partial x_k} L_{b_k} = 0, \\
\frac{\partial L_{T_1}}{\partial t} + \frac{\partial (u_k L_{T_1})}{\partial x_k} = \frac{K_1}{T_1^2} \frac{\partial T_1}{\partial x_k} \frac{\partial T_1}{\partial x_k} + K_{12} \frac{T_2 - T_1}{T_1} + \frac{\mu}{T_1} \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_k}\Big(\frac{K_1}{T_1} \frac{\partial T_1}{\partial x_k}\Big), \\
\frac{\partial L_{T_2}}{\partial t} + \frac{\partial (u_k L_{T_2})}{\partial x_k} = \frac{K_2}{T_2^2} \frac{\partial T_2}{\partial x_k} \frac{\partial T_2}{\partial x_k} + K_{12} \frac{T_1 - T_2}{T_2} + \frac{\partial}{\partial x_k}\Big(\frac{K_2}{T_2} \frac{\partial T_2}{\partial x_k}\Big), \\
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_k}\Big[u_k (E + L) + u_m b_m L_{b_k} - K_1 \frac{\partial T_1}{\partial x_k} - K_2 \frac{\partial T_2}{\partial x_k} + u_i \frac{\partial}{\partial x_k}\Big(\mu \frac{\partial u_i}{\partial x_k}\Big)\Big] = 0.
\end{gathered} \tag{4.8}$$

The viscosity μ, thermal conductivities K_1 and K_2, and the coefficient of thermal relaxation K_{12} are assumed to be positive. Here we do not discuss the well-known facts concerning the consistency of the dissipative terms included in the equations, which were discussed in [1] for a similar case. A detailed description of all possible versions of account of dissipative processes in arbitrary systems of Galilean-invariant equations should be the subject of further investigation.

In order to render the system of compatible conservation laws (4.8) specific, we specify the equation of state

$$\mathcal{E} = \mathcal{E}(\rho, S_1, S_2) = \mathcal{E}^{(1)}(\rho, S_1) + \mathcal{E}^{(2)}(\rho, S_2)$$

and set

$$q_0 = \mathcal{E}(\rho, S_1, S_2) + \rho\mathcal{E}_\rho(\rho, S_1, S_2) + S_1\mathcal{E}^{(1)}_{S_1}(\rho, S_1) + S_2\mathcal{E}^{(2)}_{S_2}(\rho, S_2) - u_k u_k/2,$$

$$T_1 = \mathcal{E}^{(1)}_{S_1}, \qquad T_2 = \mathcal{E}^{(2)}_{S_2}, \qquad L = \rho^2\mathcal{E}_\rho(\rho, S_1, S_2) + b_k b_k/2.$$

In this case, it turns out that

$$L_{q_0} = \rho, \qquad L_{u_i} = \rho u_i, \qquad L_{T_j} = \rho S_j,$$

$$E = q_0 L_{q_0} + u_k L_{u_k} + T_j L_{T_j} + b_k L_{b_k} - L = \rho(\mathcal{E}(\rho, S_1, S_2) + u_k u_k/2) + b_k b_k/2$$

and equalities (4.8) are specified in the form known as the equations of two-temperature magnetic hydrodynamics (the version describing a collisionless plasma in a certain range of parameters; see [9]). Summing up two penultimate equalities in (4.8), we obtain

$$\frac{\partial[\rho(S_1 + S_2)]}{\partial t} + \frac{\partial}{\partial x_k}\Big[\rho u_k(S_1 + S_2) - \frac{K_1}{T_1}\frac{\partial T_1}{\partial x_k} - \frac{K_2}{T_2}\frac{\partial T_2}{\partial x_k}\Big]$$

$$= \frac{K_1}{T_1^2}\frac{\partial T_1}{\partial x_k}\frac{\partial T_1}{\partial x_k} + \frac{K_2}{T_2^2}\frac{\partial T_2}{\partial x_k}\frac{\partial T_2}{\partial x_k} + K_{12}\frac{(T_2 - T_1)^2}{T_2 T_1} + \frac{\mu}{T_1}\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k} \geqslant 0.$$

Thus, we arrive at the statement that can be formulated as the law of increase of total entropy.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 01-01-00766).

REFERENCES

1. S. K. Godunov and V. M. Gordienko, "The simplest Galilean-invariant and thermodynamically consistent conservation laws," *J. Appl. Mech. Tech. Phys.*, **43**, No. 1, 1–12 (2002).
2. S. K. Godunov and E. I. Romensky, "Thermodynamics, conservation laws and symmetric forms of differential equations in mechanics of continuous media," in: *Computational Fluid Dynamics Review*, John Wiley and Sons, Chichester–New York (1995), pp. 19–30.
3. S. K. Godunov and E. I. Romensky, *Elements of Continuum Mechanics and Conservation Laws* [in Russian], Nauchnaya Kniga, Novosibirsk (1998).
4. E. I. Romensky, "Hyperbolic systems of thermodynamically compatible conservation laws in continuum mechanics," *Math. Comput. Model.*, **28**, No. 10, 115–130 (1998).
5. V. M. Gordienko, "Matrix entries of real representations of the groups $O(3)$ and $SO(3)$," *Sib. Math. J.*, **43**, No. 1, 36–46 (2002).
6. I. E. Tamm, *Foundations of the Theory of Electricity* [in Russian], Nauka, Moscow (1976).
7. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics*, Vol. 8: *Electrodynamics of Continuous Media*, Pergamon Press, Oxford-Elmsford, New York (1984).
8. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics*, Vol. 6: *Fluid Mechanics*, Pergamon Press, Oxford-Elmsford, New York (1987).
9. V. S. Imshennik and N. A. Bobrova, *The Dynamics of a Collisional Plasma* [in Russian], Énergoatomizdat, Moscow (1997).